LATTICE FIELD THEORY 1

Motivation (B physics)

QCD

Numerical Methods in QM

Motivation (Recent Breakthrough)

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Motivations for Lattice Field Theory

Lattice field theory is a rigorous way to define quantum field theory, perhaps the only way. Those who aim to "construct" quantum field theory start with a lattice. Their problem then starts out mathematically well-defined (see below), and they try (with the renormalization group) to maintain control over the continuum limit.

Field theory on a lattice is formally the same as classical statistical mechanics. Thus, it provides a new toolkit to carry out practical calculations. *E.g.*, at long distances perturbation theory (the high-energy theorist's favorite tool) breaks down for quantum chromodynamics (QCD). (Similarly the Standard Higgs sector at short distances.)

Indeed there are several problems in high-energy physics, nuclear physics, and astrophysics where non-perturbative information from QCD is needed. Lattice QCD calculations give matrix elements in B decays, information on proton structure, and the equation of state as the universe cools from a quark-gluon soup to hadrons.

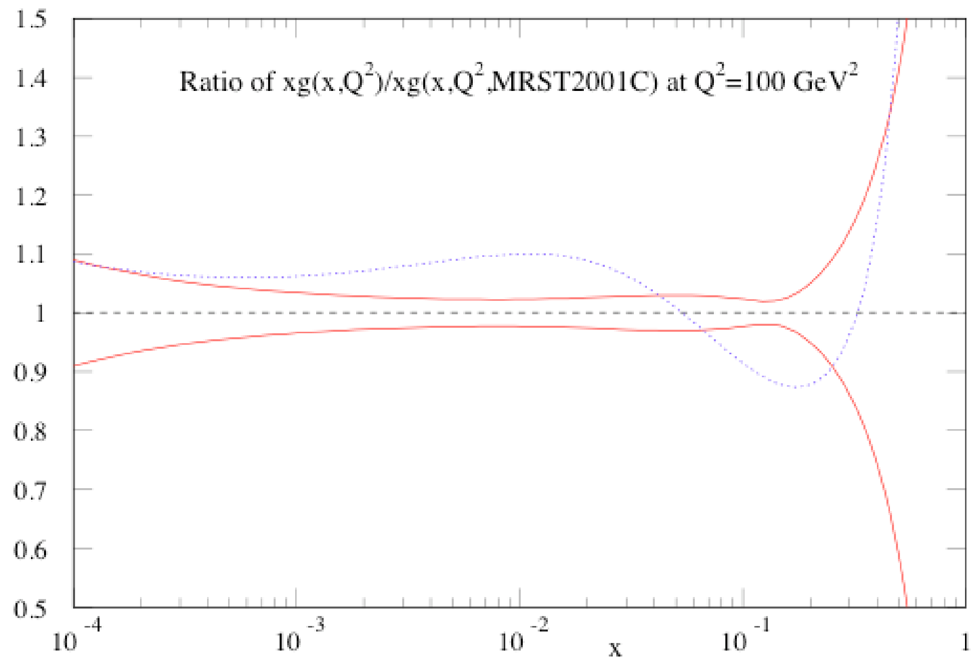
Flavor and Hadron-Collider Physics

There are two places in particle physics where lattice calculations are (or will be) especially important: quark flavor physics and hadron-hadron collisions. Both will help us infer whether there are new phenomena at play in experiments.

At the right we have a plot of the theoretical uncertainty in the gluon density of the proton vs. x.

High mass particles need large x partons and must be seen above background.

Moments of such functions can be calculated in lattice QCD, thereby constraining the large x behavior.



Flavor Physics

Lattice QCD calculations will play a more obvious role in quark flavor physics.

Fascinating effects arise from interactions between quarks and the Higgs field.

The Higgs-quark interactions generate quark masses.

They also generate flavor violation: the charged $W^{\pm}$ bosons couple all up-type quarks to all down-type quarks.

In in the case of 3 or more generations of quarks, it also generates CP violation, a necessary ingeredient for generating a baryon asymmetry.

This mechanism is the source of CP violation in the Standard Model. Standard $C\!\!\not\!P$ is probably not enough, and experiments are for other sources.

Let us review how this comes about in the Standard $SU(2) \times U(1)_Y$ Model.

Right-Handed Interactions

The right-handed fermions do not couple to $W^{\pm}$, and they are singlets under $SU(2)$:

$$U_R = (u_R, c_R, t_R)\,, \qquad Y_U = \frac{2}{3};$$
$$D_R = (d_R, s_R, b_R)\,, \qquad Y_D = -\frac{1}{3};$$

where the hypercharge Y is given.

The gauge and kinetic interactions for G generations of such fields are

$$\mathcal{L}_R = \sum_{i=1}^{G} \bar{E}^i_R(i\partial\!\!\!/ - g_1 Y_E B\!\!\!/)E^i_R + \bar{D}^i_R(iD\!\!\!\!/ - g_1 Y_D B\!\!\!/)D^i_R + \bar{U}^i_R(iD\!\!\!\!/ - g_1 Y_U B\!\!\!/)U^i_R\,,$$

where B is the gauge boson of $U(1)_Y$, with coupling g_1, and D^μ is the covariant derivative of QCD: quarks are triplets under color $SU(3)$.

Left-Handed Interactions

Left-handed fermions do couple to $W^{\pm}$; they are doublets under $SU(2)$:

$$Q_L = \left(\begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L \right), \qquad Y_Q = \frac{1}{6}.$$

The $SU(2)$ quantum number is called weak isospin, and the third component I_3 distinguishes upper and lower entries.

The gauge and kinetic interactions for G generations of such fields are

$$\mathcal{L}_L = \sum_{i=1}^{G} \bar{L}_L^i (i\partial\!\!\!/ - g_1 Y_L B\!\!\!/ - g_2 W\!\!\!\!/\,) L_L^i + \bar{Q}_L^i (iD\!\!\!\!/ - g_1 Y_Q B\!\!\!/ - g_2 W\!\!\!\!/\,) Q_L^i\,,$$

where $W = W^a \sigma_a/2$ are the gauge bosons of $SU(2)$, with gauge coupling g_2.

Note that as far as gauge interactions are concerned, the generations are simply copies of each other, with a $U(G)^3$ symmetry.

Yukawa Interactions

So far these are Laws of Nature, but they are incomplete, because there are now masses: $\mathcal{L}_m = -m\left(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R\right)$, for $\psi \in \{e, \mu, \tau, d, s, b, u, c, t\}$.

Gauge invariant interactions coupling left- and right-handed fermions need at least one additional field. The Standard choice is

$$\mathcal{L}_Y = -\sum_{i,j=1}^{G}\left[\hat{y}^e_{ij}\bar{L}^i_L\phi E^j_R + \hat{y}^d_{ij}\bar{Q}^i_L\phi D^j_R + \hat{y}^u_{ij}\bar{Q}^i_L\tilde{\phi}U^j_R + \text{h.c.}\right],$$

The quantum numbers of ϕ must preserve $SU(2)$ invariance:

$$\phi = \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}, \quad Y_Q = \frac{1}{2}; \qquad\qquad \tilde{\phi} \equiv i\sigma_2\phi^* = \begin{pmatrix}\overline{\phi^0}\\ -\phi^-\end{pmatrix}, \quad Y_Q = -\frac{1}{2}.$$

The superscripts denote the electric charge $Q = Y + I_3$.

Verify that the hypercharge assignments are right.

Higgs Interactions

The scalar Higgs field has these self-interactions:

$$V(\phi) = -\lambda v^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2,$$

with two new parameters, v and λ.

Since $v^2 > 0$, $V(\phi)$ takes the shape of a sombrero with minima at all

$$\langle \phi \rangle = e^{i\langle \xi^a \rangle \sigma_a / 2v} \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}.$$

The radial components $\langle \xi^a \rangle$ can be gauged away.

For G generations, the Yukawa matrices are complex $G \times G$ matrices. At first glance, each matrix $\hat{y}^a$ seems to introduce $2G^2$ parameters: G^2 that are real and CP conserving, and another G^2 that are imaginary and, thus, CP violating.

But the residual symmetry reduces the number of physical parameters.

The non-Yukawa Lagrangian is invariant under

$$\begin{aligned} D_R &\mapsto R_d D_R\,, & \bar{D}_R &\mapsto \bar{D}_R R_d^\dagger\,, \\ U_R &\mapsto R_u U_R\,, & \bar{U}_R &\mapsto \bar{U}_R R_u^\dagger\,, \\ Q_L &\mapsto S_u Q_L\,, & \bar{Q}_L &\mapsto \bar{Q}_L S_u^\dagger\,. \end{aligned}$$

Pick S_u and R_u so that $S_d^\dagger \hat{y}^u R_u = y^u$ is diagonal, real, and non-negative. The combination $S_u^\dagger \hat{y}^d R_d$ is neither real nor diagonal. Instead

$$S_u^\dagger \hat{y}^d R_d = V y^d\,,$$

where $y^d = S_d^\dagger \hat{y}^d R_d$ is diagonal, real, and non-negative.

Cabibbo-Kobayashi-Maskawa Matrix

In this process a matrix $V = S_u^\dagger S_d$ appeared that has physical consequences. It is the Cabibbo-Kobayashi-Maskawa (or CKM) matrix.

CKM is a $G \times G$ unitary matrix. The quarks' Yukawa interactions now read

$$\mathcal{L}_{Yq} = -\sum_{i,j=1}^{G}\left[y_j^d \bar{Q}_L^i \phi V_{ij} D_R^j + \text{h.c.}\right] - \sum_{i=1}^{G}\left[y_i^u \bar{Q}_L^i \tilde{\phi} U_R^i + \text{h.c.}\right].$$

$e^{i\varphi}S_u$, $e^{i\varphi}R_u$, and $e^{i\varphi}R_d$ achieve ($\varphi \to$ baryon number.) Thus, $3G^2 - 1$ parameters out the $4G^2$ from the two arbitrary $G \times G$ matrices, leaving $G^2 + 1$.

Of these, $2G$ are in y^u and y^d, and the other $(G-1)^2$ are in the CKM matrix V.

$\frac{1}{2}G(G-1)$ are real, CP-conserving parameters;
$\frac{1}{2}(G-1)(G-2)$ are imaginary, CP violating parameters.

Introducing

$$Q_L = \begin{pmatrix} U_L \\ VD_L \end{pmatrix},$$

which diagonalizes the mass terms for the down-like quarks, it is easy to show that

$$m_k^a = \frac{v}{\sqrt{2}} y_{kk}^a,$$

for $k = 1, 2, 3$, and $a = d, u$.

In this basis the CKM matrix migrates to the charged-current vertex:

$$\mathcal{L}_{\bar{U}WD} = -\frac{g_2}{\sqrt{2}} \left[\bar{U}_L \not{W}^+ V D_L + \bar{D}_L V^\dagger \not{W}^- U_L \right],$$

where $W^\pm = (W^1 \mp iW^2)/\sqrt{2}$.

This basis, with diagonal mass matrices and the CKM matrix in the charged currents of quarks, is usually adopted in phenomenology.

Electroweak Decays

The CKM matrix is measured through flavor-changing decays of mesons (and baryons).

Since hadrons are involved, “measurements” of CKM always entail long-distance QCD properties: $\Longrightarrow$ non-perturbative $\Longrightarrow$ lattice QCD.

There are also many weak processes that occur only in (electrowek) loop diagrams: mixing, “penguin decays,” etc.

Here there may be competition for the Standard loops: *e.g.*, from superpartners.

These processes are used to probe whether there are (observable) non-Standard sources of CP violation.

Quantum Chromodynamics

Quantum chromodynamics (QCD) is the modern theory of the strong interactions: the force that binds quarks and gluons into hadrons, and, in the end, nuclear physics.

“QCD is easily described.” The Lagrangian has $1+n_f$ free parameters:

$$\mathcal{L}_{\mathrm{QCD}} = \frac{1}{2g^2}\,\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu} - \sum_f \bar{q}_f(\not{D}+m_f)q_f,$$

with gauge coupling (or strong coupling) g^2 and quark masses m_f.

SU(3): $F_{\mu\nu} = [D_\mu, D_\nu],\ D_\mu = \partial_\mu + A_\mu^a t^a$

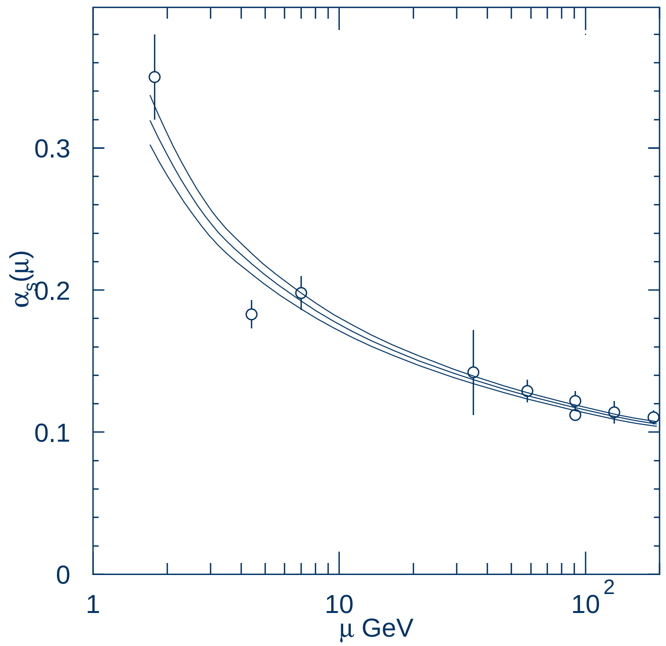
t^a_{ij}: 3×3 traceless anti-Hermitian matrices, $t^{a\dagger} = -t^a$.

q^j_f: transform und 3 representation of SU(3).

Set the parameters with $1+n_f$ experimental measurements; predict everything else.

Asymptotic Freedom

Renormalized coupling $g^2(Q)$ decreases as the momentum scale Q increases:



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$\Rightarrow$ perturbative QCD at short-distances/high energies.

Renormalized QCD remains logically sound at all energies. Cf. QED or SM Higgs.

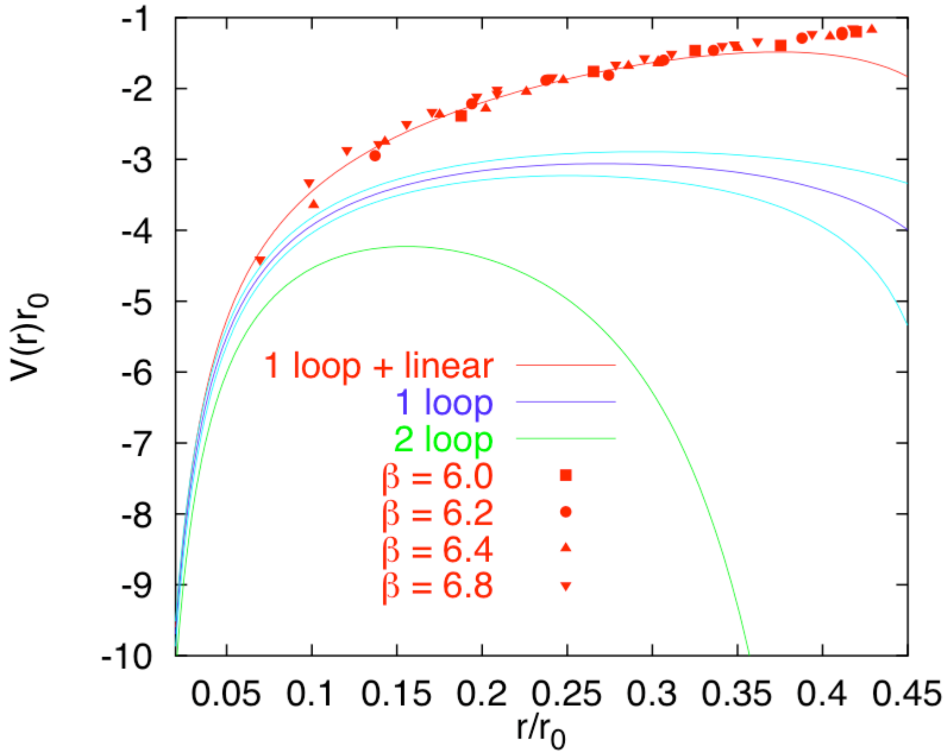
Long Distances—Color Singlets

At long distances QCD does not break down.

Perturbation theory does. $\Longrightarrow$

Quarks and gluons are confined into color singlets:

mesons $\bar{q}^i_f q^i_g$,
baryons $\varepsilon_{ijk} q^i_f q^j_g q^k_h$,
glueballs FF and FFF,
hybrids $\bar{q}qF$,
deuterons, etc.



Non-perturbative methods needed to understand long-distance QCD.

Non-Perturbative Tools

There are some general purpose tools: unitarity, analyticity, symmetry . . . .

Renormalization group tools: separate long- and short-distance dynamics, solve each part separately. Or solve one part and take the other from experiment.

Three decades ago, Kenneth Wilson returned from a scientific excursion into condensed matter physics. He had taken ideas of renormalization field theory as gifts, and returned with their tool-kit, including strong coupling expansions.

These tools exploited the formal similarity between the functional integral of quantum field theory and the partition function of statistical mechanics.

Fields on a lattice are obvious in crystals. The trick was to do the same for gauge theories such as QCD.

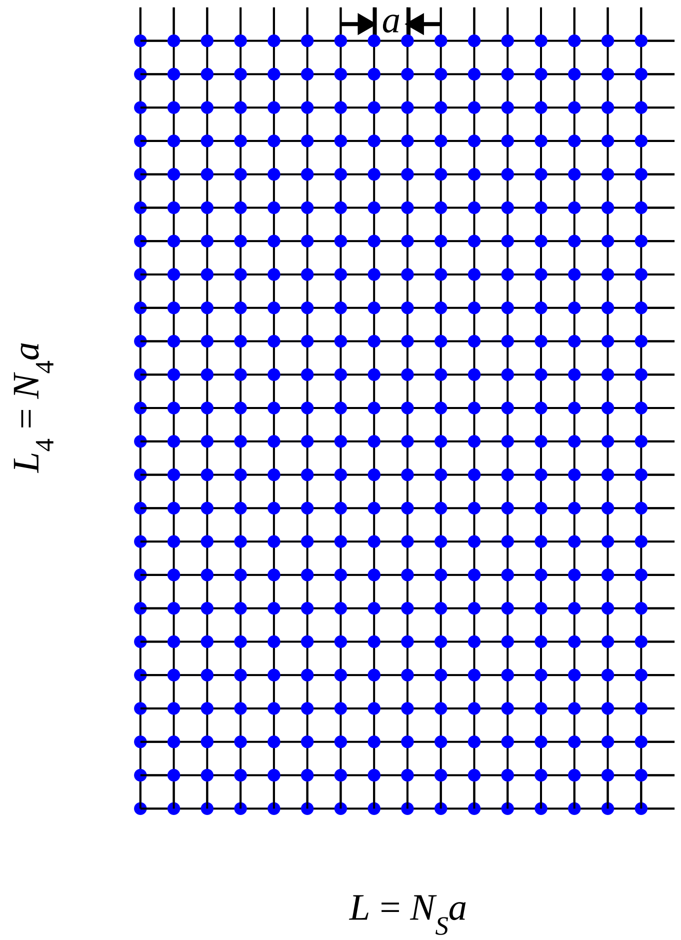
Lattice Field Theory $\longrightarrow$ Lattice Gauge Theory

Lattice field theory (*e.g.*, for a magnet) associates the degrees of freedom with the sites, links, etc., of a lattice.

For particle physics it is a (Euclidean) space-time lattice.

Wilson's idea was that the lattice fields represent aggregate degrees of freedom of the neighborhood of the sites (etc.).

Mathematical advantages: lattice provides ultraviolet cutoff from the outset; functional integral is well-defined.



Lattice Gauge Theory

The lattice breaks symmetries—most obviously space-time symmetries such as Lorentz (or Euclidean) invariance. $a^2\bar{\psi}\gamma^\mu\partial_\mu^3\psi$ suppressed

It is possible to maintain exact gauge invariance. $m_g^2 A_\mu^a A_\mu^a$ forbidden

The key is to find a gauge-covariant difference operator to play the role of $\bar{\psi}\not{D}\psi$.

The derivative compares the field at point x and (nearby) points y, such as $y = x + ae^{(\mu)}$.

$$\psi(y) \to g(y)\psi(y), \qquad \bar{\psi}(x) \to \bar{\psi}(x)g^{-1}(x)$$

Suppose we had an object that transforms as

$$U(x,y) \mapsto g(x)U(x,y)g^{-1}(y),$$

then $\bar{\psi}(x)U(x,y)\psi(y)$ is gauge-invariant.

From continuum gauge-field theory, we have a suitable object

$$U(x,y) = \mathsf{P}\exp\left(\int_y^x dz\cdot A\right),$$

ordered along some path from y to x. Verify $U(x,y) \mapsto g(x)U(x,y)g^{-1}(y)$.

These $U(x,y)$s are often called parallel transporters (by mathematicians) or a Wilson lines (by physicists).

To maintain gauge invariance, the Lagrangian of lattice gauge theory must be built out of combinations

$$\bar{\psi}(x)U(x,y)\psi(y), \qquad \mathrm{tr}[U(z,z)],$$

with the parallel transport along (closed) paths on the lattice.

There is no “best” combination. In subsequent lectures, we will discuss several different Lagrangians for lattice QCD, balancing simplicity, theoretical requirements, and reduced discretization errors.

Numerical Methods

Most lattice QCD is numerical integration of the functional integral.

The numerical work mystifies many people, but it is easy to learn the basic concepts using quantum mechanics, as (I hope) you've seen in the tutorial.

Consider the propagator in quantum mechanics, with Hamiltonian $H = p^2/2m + V(x)$:

$$\langle x(T)|x(0)\rangle = \langle x_T|e^{i\hat{H}T}|x_0\rangle = \sum_n \langle x_T|n\rangle e^{iE_nT}\langle n|x_0\rangle,$$

To derive the path integral, split the time T into many little intervals $\delta = T/N$. Then

$$\langle x_T|e^{-i\hat{H}T}|x_0\rangle = \int \prod_{i=1}^{N-1} dx_i \prod_{i=0}^{N-1} \langle x_{i+1}|e^{-i\hat{H}\delta}|x_i\rangle,$$

repeatedly inserting $1 = \int dx_i |x_i\rangle\langle x_i|$.

We would like to derive an expression for

$$\langle x_{i+1}|e^{-i\hat{H}\delta}|x_i\rangle \approx \langle x_{i+1}|e^{-i\hat{V}(x)\delta/2}e^{-i\hat{p}^2\delta/2m}e^{-i\hat{V}(x)\delta/2}|x_i\rangle = e^{-iV(x_{i+1})\delta/2}\langle x_{i+1}|e^{-i\hat{p}^2\delta/2m}|x_i\rangle e^{-iV(x_i)\delta/2}.$$

With analysis, this is possible through analytical continuation

$$\langle x_{i+1}|e^{-\hat{p}^2 a/2m}|x_i\rangle = \sqrt{\frac{m}{2\pi a}}e^{-m(x_{i+1}-x_i)^2/2a}.$$

with $a = \varepsilon + i\delta$, $\varepsilon \to 0^+$.

For numerical work, the analytical continuation is not feasible. We simply make do with propagation through imaginary time.

This “Euclidean field theory” is common in mathematical physics, for the same reason. It is better to work with well-defined expressions, and continue as a last resort.

Then one has (for imaginary time, $T \to -iT$)

$$\langle x_T | e^{-\hat{H}T} | x_0 \rangle = \lim_{N\to\infty} \int \mathcal{D}x \, \exp\left(a \sum_{i=0}^{N-1} L_i \right), \qquad \mathcal{D}x = \prod_{i=1}^{N-1} dx_i \sqrt{\frac{m}{2\pi a}},$$

where the limit is taken with T fixed, $a = T/N$.

The (discrete time) Lagrangian is

$$L_i = -\tfrac{1}{2} m \left(\frac{x_{i+1} - x_i}{a} \right)^2 - \tfrac{1}{2} V(x_{i+1}) - \tfrac{1}{2} V(x_i),$$

which one recognizes as a discrete approximation to the kinetic energy and the average of the potential energy over two times.

In numerical work, one uses a sequence of Ns and extrapolates.

As we shall discuss in subsequent lectures, one has theoretical control over the extrapolation, even for field theory where issues of renormalization must be addressed.

Old Pessimism $\longrightarrow$ New Optimism!

It is a good time to take up the study of lattice QCD.

To reduce the computational burden, until now almost all calculations of physically interesting masses or hadronic matrix elements have been done in the so-called "quenched approximation."

This corresponds to omitting all vacuum loops, and compensating the omission with *ad hoc* shifts in the bare gauge coupling and masses.

It's a bit like a dielectric approximation, $e^2 \to e^2/\varepsilon$. One can only hope that it works when focusing on a narrow range of energies [$\varepsilon(\omega) = \text{constant}$].

For example, many (of my own) papers include statements like "the nth error bar comes from the discrepancy in determining m_b from Υ spectrum instead of the B system."

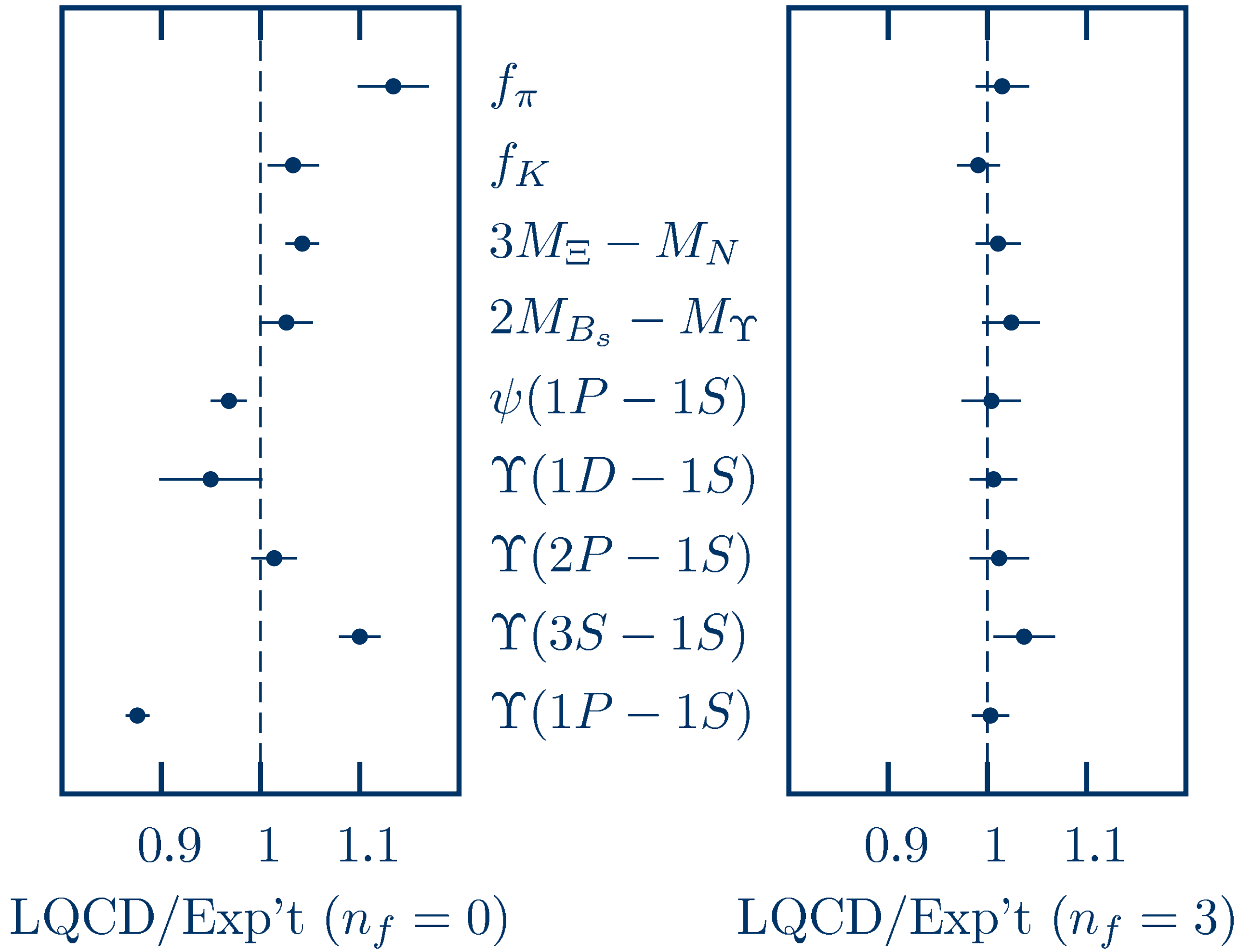
'Twas very unsatisfactory.

This year, however, 26 authors produced this plot: $\Longrightarrow$

Set $1+4$ free parameters with $1+4$ meson masses.

Quenched (on left) shows discrepancies as much as 10–15%.

Unquenched QCD (on right) shows discrepancies of a few %—within the error bars.



Davies *et al.*, hep-lat/0304004

The five fiducial quantities ($m_{\Upsilon(2S)} - m_{\Upsilon(1S)}$, m_π^2, m_K^2, m_{D_s}, and $m_{\Upsilon(1S)}$) and the nine shown are all, in a certain sense, "gold-plated."

The gold-plated class includes stable-particle masses and hadronic matrix elements with at most one hadron in the initial or final states

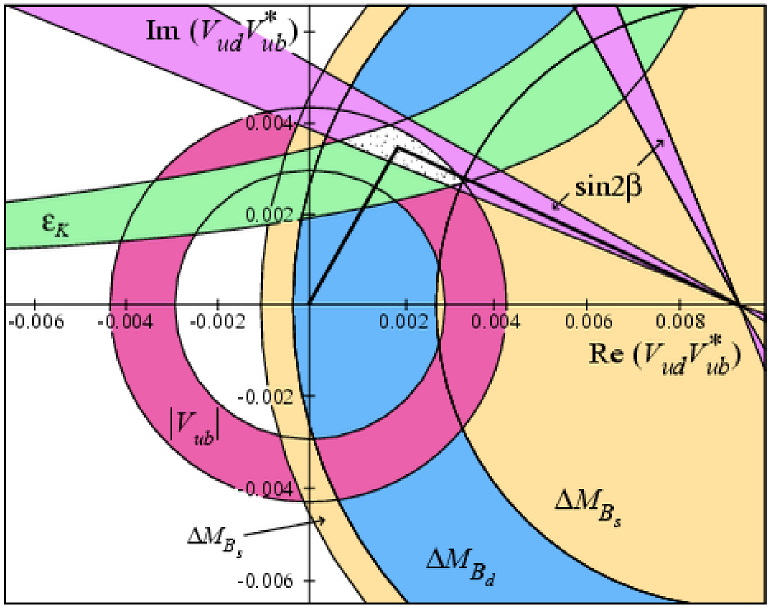
Unstable particles and non-leptonic decays inevitably entail multi-particle states—much more difficult (to be explained later).

This may seem like a disappointing restriction.

There are, however, gold-plated matrix elements for extracting *all* CKM elements $|V_{qq'}|$, except $|V_{tb}|$. (Top quark decays before hadronizing.)

It's not unrealistic to expect the theoretical uncertainty in the CKM matrix to be reduced to a few percent in the next 2–3 years.

The (CKM) Matrix Reloaded



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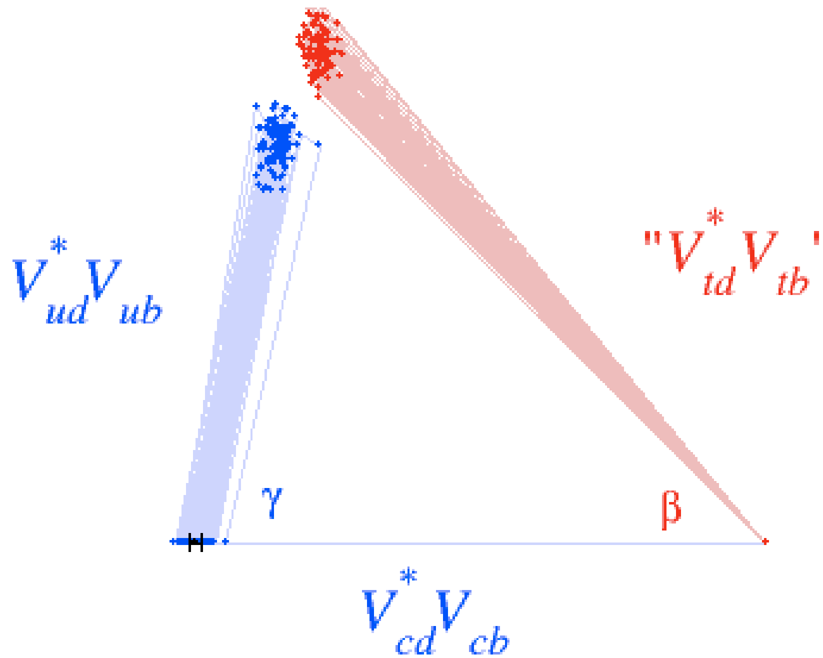
theoretical uncertainties dominate



reduce theoretical uncertainties to a few %

reduce also exp'tal uncertainty in $|V_{ub}|$